Stochastic Gravitational Wave Background in Brane World Cosmology

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(Dated: February 2, 2008)

We investigate the cosmological evolution of gravitational waves in Friedman-Robertson-Walker brane world models embedded in a five dimensional anti de-Sitter spacetime. To predict the spectrum of stochastic gravitational background at present generated in the inflationary phase, we numerically calculate the evolution of gravitational waves according to the method developed in our previous paper [K. Ichiki and K. Nakamura, arXiv:hep-th/0310282]. The resulting logarithmic energy spectrum is significantly altered from that of the standard four dimensional inflationary models in the frequency range $f \gtrsim f_{\rm AdS} \approx 10^{-4}$ Hz, which corresponds to the curvature scale we set of anti de-Sitter spacetime, $l \approx 1$mm.

In recent years much attention has been paid to interesting models of spacetimes in which fields of the standard model are confined to a "brane", while only the gravitational field propagates in all the dimensions called the "bulk". Since the proposal of a brane world model of our spacetime by Randall and Sundrum [1] (RS II model), the phenomenology of brane world cosmological models has been the subject of intensive investigations [2]. In their models our four dimensional spacetime is embedded in the five dimensional anti de-Sitter (AdS5) spacetime with curvature scale l, and the four-dimensional gravity is effectively recovered in larger scales than l. The current direct bound of the experimental gravitational probe allows the scale of extra dimensions or curvature scale l to be as large as a (sub)millimeter [3].

In addition to the direct experimental probe of the Newton potential, if our universe is really the RS II type brane universe, there is a possibility that the evolution history of the universe is significantly altered when the Hubble scale H^{-1} is smaller than l in the early universe. Actually, some authors have been discussed new physics in the epoch of the early universe where Hubble scale is smaller than l and the possible sources of gravitational waves[4, 5]. In particular, it is pointed out that these new physics in the early universe will appear in the stochastic gravitational wave background with the characteristic frequency around 10^{-4} Hz[4], if the scale of l is order of 1mm.

Thus, the stochastic gravitational waves will be a promising candidate which provides direct and deep probes of the early universe (for example, see [6]). One might see in principle earlier epochs of the universe than the photon last scattering surface by gravitational waves.

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In order to give theoretical predictions of the stochastic gravitational waves, we have recently proposed the single null coordinate system to solve the gravitational waves in the expanding brane universe[7], based on the idea in Ref.[8]. We have concentrated on the brane world model of the flat Friedmann-Robertson-Walker (FRW) universe without "dark radiation" following discussions in Ref.[9] and the fact that the flat FRW universe is supported by recent precise measurements of the cosmic microwave background (CMB) anisotropies [10]. Though some authors have investigated based on the Gaussian normal coordinate system in the neighborhood of the brane [11], this coordinate system includes a coordinate singularity in the bulk, where the metric function vanishes [12], and the treatment of wave equation near the singularity becomes delicate. However, these delicate problems are removed by the introduction of the coordinate system proposed in Ref.[7].

In this letter, we investigate how the presence of extra dimension of RS type leaves an imprint on the stochastic gravitational wave background at present. We solve the evolution of gravitational waves in the brane world model and predict the present energy spectrum, following the previous proposal[7]. In terms of the single null coordinate system proposed in [7], the metric on the AdS_5 is given by

$$ds^2 = -F(\tau,\overline{w})du^2 - 2F(\tau,\overline{w})du d\overline{w} + r^2(\tau,\overline{w})d\Sigma_0^2 \ . \quad (1)$$

Here the metric functions $F(\tau,\overline{w})$ and $r(\tau,\overline{w})$ are given by

$$F(\tau(u,\overline{w}),\overline{w}) = \frac{r^2/l^2}{\sqrt{r^2/l^2+\dot{a}^2}+\dot{a}} \ , \quad (2)$$

$$r(\tau(u,\overline{w}),\overline{w}) = \frac{a(\tau)}{2} \times \left[(1-A)e^{\overline{w}/l} + (1+A)e^{-\overline{w}/l}\right] \ ,$$

respectively, where $A \equiv \sqrt{1+l^2H^2}$. Here $a(\tau)$ is scale factor of the FRW brane whose time evolution is given

by the generalized Friedmann equation [2]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_{\rm N}}{3}\rho + \frac{\Lambda_4}{3} + \frac{\kappa_5^4}{36}\rho^2. \qquad (3)$$

In Eq.(3), $\dot{a} = da/d\tau$, τ is the comoving time on the FRW brane, ρ is the energy density on the brane, $\Lambda_4 = \kappa_5^4\lambda^2/12 + 3\Lambda_5/4$ is the cosmological constant induced on the brane, $G_{\rm N} = \kappa_5^4\lambda/48\pi$ and κ_5 are the four-dimensional and five-dimensional gravitational constants, respectively. We also assume Z_2 symmetry across the brane following to the original RS II model. Hereafter we set $l = 1$ mm for the purpose of numerical calculations.

Apart from the polarization tensor, the equation for gravitational waves in the bulk is simply given by that for the five dimensional massless scalar field, $\Box_5 h = 0$. As derived in Ref.[7], this equation is equivalent to:

$$\begin{aligned} \partial_u h &= r^{-3/2}\left[g\partial_{\overline{w}}h\right]_0^{\overline{w}} + r^{-3/2} \qquad (4)\\ &\times \left\{\int_0^{\overline{w}}\left(\partial_{\overline{w}}h\left(J - \frac{\partial g}{\partial \overline{w}}\right) - g\frac{k^2}{r^2}h\right)dw + C(u)\right\}, \end{aligned}$$

where $-k^2$ is the eigen value of the Laplacian of Σ_0, and

$$g := \frac{F}{2}r^{3/2}, \qquad (5)$$

$$J := \left(\frac{1}{2}\frac{\partial F}{\partial \overline{w}} + \frac{3F}{2}\frac{\partial \ln r}{\partial \overline{w}} - \frac{3}{2}\frac{\partial \ln r}{\partial u}\right)r^{3/2}. \qquad (6)$$

$C(u)$ is determined by the boundary conditions of the Neumann type on the brane at each time step

$$C(u) = a^{3/2}F(\tau, w) \times \partial_{\overline{w}}h(u, \overline{w})|_{\overline{w}=0}. \qquad (7)$$

Once given an initial spectrum on an initial null hypersurface in the bulk, which is a $u = const$ hypersurface in the bulk, Eqs.(3)-(7) are enough to predict the evolution of gravitational waves.

As an initial condition of numerical examples, we follow the inflationary scenario and we start numerical calculation by setting $h = const.$ on a initial null hypersurface. This initial value is a solution to the wave equation in the long wavelength limit $k/aH \to 0$ [13]. We should note, however, that, for $k \neq 0$ mode, $h = const.$ is not an exact solution to the wave equation but an approximate one in the situation $k/aH \ll 1$. Further, as the universe expands and decelerates, the approximation $k/aH \ll 1$ becomes violated. Conversely, this also implies that $h = const.$ is an appropriate initial condition in the long wavelength approximation, if we start the calculation from the past much earlier time than that of the horizon crossing of each mode on the brane. For this reason, we start the calculation when the wave is far enough outside the horizon with the initial condition $h = const.$ to keep the validity of the long wavelength approximation $k/aH \to 0$.

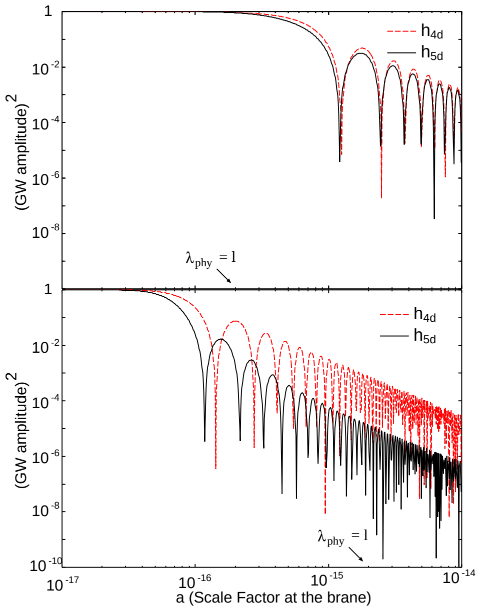

FIG. 1: The time evolution of gravitational waves with comoving frequency, $f = 5\times10^{-5}$[Hz] (upper panel) and $f = 5\times10^{-4}$ [Hz] (lower panel). The time evolution of gravitational wave with $f = 5\times10^{-5}$[Hz] is almost "four dimensional" one, in which amplitude of gravitational waves scales as $h_{4D} \propto a^{-1}$. On the other hand, the amplitude of gravitational wave with $f = 5\times10^{-4}$ [Hz] in brane world model decreases faster than a^{-1} when $\lambda_{\rm phy} \lesssim l$. After that, its evolution asymptotically becomes similar to that of standard model when $\lambda_{\rm phy} \gtrsim l$.

We show the examples of the evolution of gravitational waves $h_{5\rm d}$ in Fig.1, which are obtained by solving Eqs.(3)-(7), numerically. In these figures we also show "four dimensional gravitational waves" $h_{4\rm d}$ for comparison, which is a solution of the four dimensional equation $\ddot{h}_{4\rm d} + 3H\dot{h}_{4\rm d} + (k^2/a^2)\,h_{4\rm d} = 0$ and simply evolves as a^{-1}.

The feature of the evolutions of gravitational waves, which one can easily see from Fig.1, is that the amplitudes of them are constant in time at first, and then start damping with oscillation. In the standard four-dimensional cosmology, it is well known that the amplitude of gravitational waves is "frozen" when the modes are outside the horizon ($k/aH \ll 1$), and then starts to oscillate after the modes enter the horizon ($k/aH \gg 1$). As shown in Fig.1, it turns out that this behavior is also seen in the evolution of gravitational waves in brane world model considered here.

The other important feature is that the amplitude of five-dimensional gravitational waves $h_{5\rm d}$ decreases faster than that of standard four-dimensional gravitational waves $h_{4\rm d}$ in higher frequency modes (the lower panel in Fig.1). This is an essential feature of the five-dimensional evolution of gravitational waves. As we mentioned above, the gravitational waves with wavelength smaller than the curvature scale of AdS$_5$ ($\lambda_{\rm phy} \equiv 2\pi a/k \lesssim l$) would *feel*

the universe to be five dimensional, and thus the evolution of gravitational waves with such wavelength can be significantly altered. Even though the decomposition of independent Kaluza-Klein (KK) modes is quite nontrivial in FRW brane universe, the strongly decelerating brane motion and Doppler effects by moving brane cause the mode mixing of gravitational waves, and do generate the effective "KK modes" which propagate into the bulk [11]. This is the reason for the damping in amplitude of five-dimensional gravitational waves relative to that of four-dimensional gravitational waves.

In Fig.2, we show both the important criterions discussed above. We depict the scale factors when each Fourier mode with comoving frequency $f(=k/2\pi)$ enters the Hubble horizon ($\lambda_{\rm phy}=1/H$; line), and when their wavelength becomes longer than the curvature scale of AdS_5 ($\lambda_{\rm phys}=l$; dashed line) in Fig.2. A break around $f\approx 5\times 10^{-4}$ Hz in the line $\lambda_{\rm phy}=1/H$ corresponds to the frequency of the mode which crosses the Hubble horizon at the transition from ρ^2-term dominate era to the standard radiation dominate era (see, Eq. (3)). These two lines cross each other ($\lambda_{\rm phy}=1/H=l$) at the comoving frequency $f_{\rm AdS}\approx 10^{-4}$ Hz. Figure 2 is helpful to understand that gravitational waves with comoving frequency larger than $f_{\rm AdS}$ Hz are affected by the existence of the fifth dimension, and the gravitational waves change their behavior at $f=f_{\rm AdS}$

To explain this, let us first consider the lower frequency modes, $f\lesssim f_{\rm AdS}$. Figure 2 shows that the physical wavelength $\lambda_{\rm phy}$ is larger than l first, but the mode is still far outside the horizon. After that, as the universe expands, the mode crosses the horizon and starts to oscillate. Since the physical wavelength is long enough relative to the curvature length of AdS_5 at that time, the evolution of gravitational waves is effectively four-dimensional one (see the upper panel in Fig.1).

On the other hand, the gravitational waves with comoving frequency greater than $f_{\rm AdS}$ enter the Hubble horizon, first. Then, they start to oscillate and their amplitude decreases faster than a^{-1} due to the "KK mode" mixing discussed above. After that, as the universe expands the wavelength of gravitational waves finally becomes longer than the curvature scale l of AdS_5, and the time evolution of gravitational waves asymptotically mimics that of the standard four-dimensional FRW model, i.e., the amplitude evolves as $h_{\rm 5d}\propto a^{-1}$. The evolution of gravitational waves in the lower panel of Fig.1 does show this behavior.

In order to predict the spectrum of stochastic gravitational waves at present, we have to know how much amplitude of gravitational waves in five dimensional model has escaped from the brane compared to that in four dimensional model. To accomplish this, we draw two envelope curves of the evolution of gravitational waves for $h_{\rm 5d}$ and $h_{\rm 4d}$, and numerically observe the ratio $h_{\rm 5d}/h_{\rm 4d}$ after the evolution of gravitational waves in five dimensional model exhibits asymptotic behavior $h\propto a^{-1}$. The result is depicted in Fig.3. Although there seem to exist some small scatters, we find that the ratio can be well fitted by a simple power law above some critical frequency:

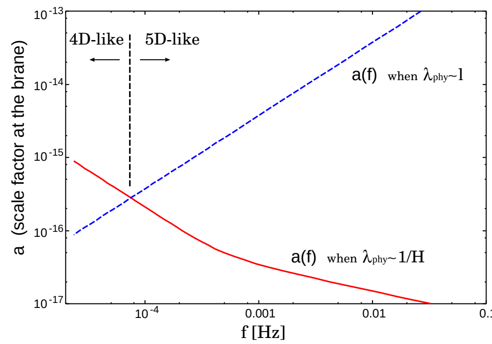

FIG. 2: Cosmic scale factors at the brane as a function of comoving frequency when each Fourier mode re-enter the Hubble horizon ($\lambda_{\rm phy}=1/H$; line) and when the wavelength becomes longer than the AdS5 curvature scale ($\lambda_{\rm phy}=l$; dashed line).

$$\frac{h_{\rm 5d}}{h_{\rm 4d}}=\begin{cases}1 & \text{for } f<f_{\rm AdS}\\ \left(\frac{f}{f_{\rm AdS}}\right)^{-\alpha} & \text{for } f>f_{\rm AdS}\end{cases} \tag{8}$$

where we can obtain from Fig.3 that $f_{\rm AdS}\approx 10^{-4}$ Hz and α is roughly about 0.9.

The spectrum of stochastic gravitational waves can be obtained by multiplying the expected amplitude of four dimensional gravitational waves today, $h_{\rm 4d}(\tau_0)$, by the factor $(h_{\rm 5d}/h_{\rm 4d})$ in Eq.(8). Present spectrum of four-dimensional gravitational waves $h_{\rm 4d}(\tau_0)$ can be obtained as follows. In high energy era when ρ^2 term dominates, Hubble parameter scales as $H\propto\rho_r\propto a^{-4}$ (see Eq.(3)) leading to $1/aH\propto a^3$. Using the fact that the amplitude of gravitational waves is almost constant until horizon crossing $k/aH\sim 1$ and decreases as $h_{\rm 4d}\propto a^{-1}$ after that, we obtain $h_{\rm 4d}\propto k^{-1/3}$ in high energy era. The same arguments lead that $h_{\rm 4d}\propto k^{-1}$ in standard radiation dominated era, and $h_{\rm 4d}\propto k^{-2}$ in matter dominated era [6]. By combining these considerations, we can predict the final spectrum of the stochastic gravitational waves.

In Fig.4 we show the logarithmic energy spectrum of gravitational waves, $\Omega_{\rm GW}(f)h_0^2$, where $\Omega_{\rm GW}(f)\propto k^2h^2$ is the energy density of gravitational waves per logarithmic frequency interval in critical density units, and h_0 is Hubble parameter today in units of 100 km/s/Mpc. We find that the spectrum becomes red, $\Omega_{\rm GW}h_0^2\propto f^{-0.46}$, in the frequency range $f\gtrsim f_{\rm AdS}$, contrary to the expectation that the spectrum would have a spike due to the non-standard (strongly decelerating) expansion law [14].

Finally we briefly mention the observational aspects. In standard models with CMB quadrupole anisotropies measured by COBE, the energy density of gravitational

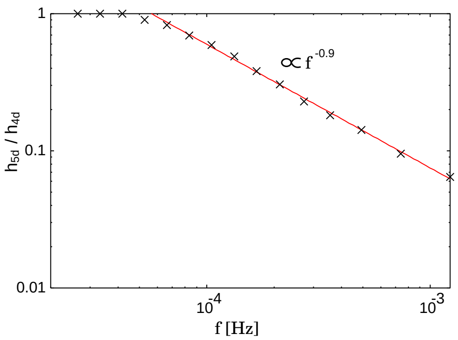

FIG. 3: The amplitude ratio of gravitational waves ($h_{\rm 5d}$) to the standard four dimensional amplitude ($h_{\rm 4d} \propto a^{-1}$) determined in the asymptotic era ($\lambda_{\rm phy} \gg l$). Damping due to the momentum into the extra dimension becomes larger with larger comoving frequency.

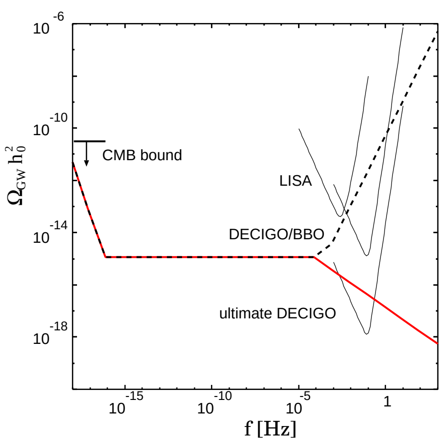

FIG. 4: The spectrum of stochastic gravitational waves in brane world model with $l = 1$ mm (line) at present. The case in which we only include the effect of non-standard expansion law is shown by dashed line for comparison. Overall amplitude is normalized by CMB bound around the frequency 10^{-18} Hz. The noise curves of LISA, practical DECIGO/BBO and quantum limited ultimate DECIGO for integration times ≈ 3 yr are also shown in the figure [17, 18, 19, 20].

waves is constrained to $\Omega_{\rm GW} \lesssim 10^{-15}$ at $f \sim 0.1$ Hz in standard cosmology. If future CMB experiments could directly measure tensor amplitude and its spectral tilt (n_T) through temperature and polarization anisotropies [15], one can in principle extrapolate and expect the amplitude of gravitational waves to much higher frequency range, even up to 10^{15} Hz [16]. On the other hand, in the brane world models considered in this paper, it is derived from our results that the spectral energy density will be much smaller than expected above some critical frequency $f_{\rm AdS} \approx 10^{-4}$ Hz. Unfortunately, stochastic radiation from white dwarf binaries prevents us from detecting inflationary primordial gravitational waves in the frequency range $10^{-5} \lesssim f \lesssim 10^{-2}$ Hz. However, the sky would be "transparent" to a primordial signal above the frequency $f \sim 0.1$ Hz by one year observation, if the neutron star - neutron star merger rate is around $R \sim 10^{-5}$/yr/galaxy [17]. In this bands, the decihertz ($10^{-2} - 10$ Hz) interferometer gravitational wave detectors, such as DECIGO [18] and BBO [19], have been proposed aiming to detect the primordial gravitational waves as one of their important goals. Therefore, the signal of such brane world cosmological models might be detected as a "lack" of gravitational waves in such higher frequencies.

K.I would like to thank M. Yahiro and R. Takahashi for helpful discussions. K.N. thanks M. Omote and S. Miyama for their continuous encouragements. K.I.'s work is supported by Grant-in-Aid for JSPS Fellows.

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